

Linear Differential Equations with Constant Coefficients

The general linear second – order equation:

$$y'' + P(x)y' + Q(x)y = R(x) \dots\dots\dots(1)$$

$$y'' + P(x)y' + Q(x)y = 0 \dots\dots\dots(2)$$

P(x), Q(x) and Q(x) are known functions

Eq.(1) is linear, 2nd order and Nonhomog.

Eq.(2) is linear, 2nd order and Homog.

Theorem1:

If y_1 and y_2 are any solutions of the homogeneous equation:

$$y'' + P(x)y' + Q(x)y = 0,$$

then $y_3 = c_1y_1 + c_2y_2$, where c_1 and c_2 are arbitrary constants, is also solution

Theorem2:

If y_1 and y_2 are two solutions of the homogeneous equation: $y'' + P(x)y' + Q(x)y = 0$, for which

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' \neq 0 \quad \text{and if}$$

$\int P(x)dx$ exists, then there exist constants c_1 & c_2 such that any solution y_3 of the homogeneous equation can be expressed in the form

$$y_3 = c_1 y_1 + c_2 y_2$$

Proof :

To prove this theorem, first show that any pair of solution of : $y'' + P(x)y' + Q(x)y = 0$, say y_i and y_j satisfies the relation:

$$W(y_i, y_j) = y_i y'_j - y_j y'_i = k_{ij} e^{-\int P(x) dx}$$

where k_{ij} is a suitable constant.

$$y_i'' + P(x)y_i' + Q(x)y_i = 0 \dots\dots\dots(1)$$

$$y_j'' + P(x)y_j' + Q(x)y_j = 0 \dots\dots\dots(2)$$

$[Eq.(2) \times y_i] - [Eq.(1) \times y_j]$ get;

$$[y_i y_j'' - y_j y_i''] + P(x)[y_i y_j' - y_j y_i'] = 0 \dots\dots(3)$$

Now:

$$\begin{aligned}\frac{dW(y_i, y_j)}{dx} &= \frac{d(y_i y'_j - y_j y'_i)}{dx} \\ &= (y_i y''_j + y'_j y'_i) - (y_j y''_i + y'_i y'_j) \\ &= (y_i y''_j - y_j y''_i)\end{aligned}$$

Hence, Eq. (3) can be written;

$$\frac{dW(y_i, y_j)}{dx} + P(x) W(y_i, y_j) = 0$$

$$\ln W(y_i, y_j) = -\int P(x) dx + c$$

$$W(y_i, y_j) = k_{ij} e^{-\int P(x) dx}$$

*Now, for two pairs of solutions
(y₃, y₁) and (y₃, y₂)*

$$y_3 y_1' - y_1 y_3' = k_{31} e^{-\int P(x) dx}$$

$$y_3 y_2' - y_2 y_3' = k_{32} e^{-\int P(x) dx}$$

*Solve, these two above equations
getting;*

$$y_3 = \frac{y_1 k_{32} e^{-\int P(x) dx} - y_2 k_{31} e^{-\int P(x) dx}}{y_1 y_2' - y_2 y_1'}$$

$$y_3 = \frac{y_1 k_{32} e^{-\int P(x) dx} - y_2 k_{31} e^{-\int P(x) dx}}{k_{12} e^{-\int P(x) dx}}$$

$$\therefore y_3 = \frac{k_{32}}{k_{12}} y_1 - \frac{k_{31}}{k_{12}} y_2$$

$$y_3 = c_1 y_1 + c_2 y_2$$

$$\text{where } c_1 = \frac{k_{32}}{k_{12}} \quad \text{and } c_2 = -\frac{k_{31}}{k_{12}}$$

Let us suppose that $y_1(x) \neq 0$ is a solution of,

$y'' + P(x)y' + Q(x)y = 0$, and let us attempt to

find a function $\Phi(x)$ with the property that

$\Phi(x)y_1(x)$ is also a solution of above equation

Now, substitute $y = \Phi(x)y_1(x)$ into above equation:

$$(y_1''\Phi + 2y_1'\Phi' + y_1'\Phi'') + P(x)(y_1'\Phi + y_1\Phi') + Q(x)(y_1\Phi) \\ = (y_1'' + P(x)y_1' + Q(x)y_1)\Phi + (2y_1' + P(x)y_1)\Phi' + y_1\Phi'' = 0$$

$$y_1\Phi'' + (2y_1' + P(x)y_1)\Phi' = 0$$

$$\frac{d\Phi'}{\Phi'} + \left[\frac{2y_1'}{y_1} + P(x) \right] dx = 0$$

$$\ln(\Phi') + 2\ln(y_1) + \int P(x)dx = \ln(c)$$

$$\ln \frac{\Phi' y_1^2}{c} = - \int P(x)dx$$

$$\ln \frac{\Phi' y_1^2}{c} = -\int P(x) dx$$

$$\Phi' = \frac{c e^{-\int P(x) dx}}{y_1^2}$$

$$\Phi = c \int \frac{e^{-\int P(x) dx}}{y_1^2} dx + k$$

For all values of c & k , the solution,

$$\Phi(x) y_1(x) = c y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2} dx + k y_1(x)$$

$$\Phi(x) y_1(x) = c y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2} dx + k y_1(x)$$

*The $\Phi(x)y_1(x)$ is also a complete solution,
since 2 constants provided*

$$W(y_1, \Phi y_1) \neq 0$$

Example:

Find a complete solution of equation

$x^2 y'' + xy' - 4y = 0$, given that $y = x^2$ is one solution

Solution:

Substitute $y = x^2 \Phi$ into the given D.E.

$$x^2 (x^2 \Phi)'' + x(x^2 \Phi)' - 4(x^2 \Phi) = 0$$

$$x^2 (2\Phi + 2x\Phi' + 2x\Phi' + x^2 \Phi'') + x(2x\Phi + x^2 \Phi') - 4(x^2 \Phi) = 0$$

$$2x^2 \Phi + 4x^3 \Phi' + x^4 \Phi'' + 2x^2 \Phi + x^3 \Phi' - 4x^2 \Phi = 0$$

$$x^3 (x\Phi'' + 5\Phi') = 0$$

$$x\Phi'' + 5\Phi' = 0$$

$$x \frac{d\Phi'}{dx} + 5\Phi' = 0$$

$$\frac{d\Phi'}{\Phi'} + 5 \frac{dx}{x} = 0$$

$$\ln \Phi' + 5 \ln x = \ln c$$

$$\Phi' = \frac{c}{x^5}$$

$$\Phi = -\frac{c}{4x^4} + k$$

The complete solution is :

$$y = x^2 \Phi = -\frac{c}{4x^2} + kx^2$$

Theorem3:

If Y is any solutions of the nonhomogeneous equation:

$$y'' + P(x)y' + Q(x)y = R(x),$$

and if $c_1y_1 + c_2y_2$ is a complete solution of the homogeneous equation obtained from this by deleting the term $R(x)$, then

$y = c_1y_1 + y_2c_2 + Y$ is a complete solution of the nonhomogeneous equation

Proof :

Let y_i be any solution of nonhomogeneous equation:

$$y_i'' + P(x)y_i' + Q(x)y_i = R(x) \dots\dots\dots(1)$$

and similarly, since Y is a solution of nonhomogeneous equation:

$$Y'' + P(x)Y' + Q(x)Y = R(x) \dots\dots\dots(2)$$

Eq.(1) – Eq.(2);

$$(y_i'' - Y'') + P(x)(y_i' - Y') + Q(x)(y_i - Y) = 0$$

OR;

$$(y_i - Y)'' + P(x)(y_i - Y)' + Q(x)(y_i - Y) = 0$$

Thus the quantity $(y_i - Y)$ satisfies the homogeneous equation and, hence, by theorem(2), must be expressible in the form;

$$y_i - Y = c_1 y_1 + c_2 y_2$$

Since y_i was any solution of the nonhomogeneous equation Theorem (3) is thus established.

The homogeneous linear equation with constant coefficients :

$$\frac{dy}{dx} = Dy, \quad \frac{d^2y}{dx^2} = D(Dy) = D^2y, \quad \frac{d^3y}{dx^3} = D^3y$$

$$(aD^2 + bD + c)\phi(x) \text{ means } aD^2\phi(x) + bD\phi(x) + c\phi(x)$$

$$(3D^2 - 10D - 8)x^2 = 3(2) - 10(2x) - 8x^2 = 6 - 20x - 8x^2$$

$$\begin{aligned} (3D + 2)(D - 4)x^2 &= (3D + 2)(2x - 4x^2) = (6 - 24x) + (4x - 8x^2) \\ &= 6 - 20x - 8x^2 \end{aligned}$$

$$\begin{aligned} (D - 4)(3D + 2)x^2 &= (D - 4)(6x + 2x^2) = (6 + 4x) - (24x + 8x^2) \\ &= 6 - 20x - 8x^2 \end{aligned}$$

$ay'' + by' + cy = 0$ can be written $(aD^2 + bD + c)y = 0$

It natural to try: $y = e^{mx}$

as a solution because all derivatives are alike

except for coefficient.

Then;

$$e^{mx}(am^2 + bm + c) = 0$$

But

$$e^{mx} \neq 0$$

So;

$$(am^2 + bm + c) = 0$$

And is called the **characteristics or auxiliary** equation of Homog. Or Nonhomog. Eq.
It is obtained as:

$$am^2 + bm + c = 0 \quad \text{or} \quad aD^2 + bD + c = 0$$

The two roots:

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The solutions:

$$y_1 = e^{m_1 x} \quad \text{and} \quad y_2 = e^{m_2 x}$$

Then:

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

By Theorem 2, this is a complete solution of Homog. equation, if

$$W(y_1, y_2) \neq 0$$

$$\begin{aligned} W(y_1, y_2) &= y_1 y_2' - y_2 y_1' \\ &= e^{m_1 x} (m_2 e^{m_2 x}) - e^{m_2 x} (m_1 e^{m_1 x}) \\ &= (m_2 - m_1) e^{(m_1 + m_2)x} \neq 0 \end{aligned}$$

Unless $m_1 = m_2$

Example 1:

$$y'' + 7y' + 12y = 0$$

Solution:

$$(D^2 + 7D + 12)y = 0$$

$$m^2 + 7m + 12 = 0$$

$$(m + 4)(m + 3) = 0$$

$$m_1 = -4 \quad \text{and} \quad m_2 = -3$$

$$\therefore y = c_1 e^{-4x} + c_2 e^{-3x}$$

Example 2:

$$y'' + 2y' + 5y = 0$$

Solution:

$$(D^2 + 2D + 5)y = 0$$

$$m^2 + 2m + 5 = 0$$

$$m^2 + 2m + 1 = -5 + 1$$

$$(m + 1)^2 = -4$$

$$m + 1 = \pm 2\sqrt{-1} = \pm 2i$$

$$m_1 = -1 + 2i \quad \text{and} \quad m_2 = -1 - 2i$$

$$m_1 = -1 + 2i \quad \text{and} \quad m_2 = -1 - 2i$$

$$y = c_1 e^{(-1+2i)x} + c_2 e^{(-1-2i)x}$$

$$y = c_1 e^{-x} e^{2ix} + c_2 e^{-x} e^{-2ix}$$

$$y = e^{-x} (c_1 e^{2ix} + c_2 e^{-2ix})$$

$$= e^{-x} [c_1 (\cos 2x + i \sin 2x) + c_2 (\cos 2x - i \sin 2x)]$$

$$= e^{-x} [(c_1 + c_2) \cos 2x + i(c_1 - c_2) \sin 2x]$$

$$y = e^{-x} [A \cos 2x + B \sin 2x]$$

Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Example 3 :

$$y'' + 6y' + 9y = 0$$

Solution:

$$(D^2 + 6D + 9)y = 0$$

$$m^2 + 6m + 9 = 0$$

$$(m + 3)^2 = 0$$

$$m_1 = m_2 = -3$$

$$\therefore y = c_1 e^{-3x} + c_2 x e^{-3x}$$

Example 4 :

$$y'' - 4y' + 4y = 0$$

$$y = 3, \quad x = 0$$

$$\text{and } y' = 4, \quad x = 0$$

Solution:

$$(D^2 - 4D + 4)y = 0$$

$$m^2 - 4m + 4 = 0$$

$$(m - 2)^2 = 0$$

$$m_1 = m_2 = 2$$

$$\therefore y = c_1 e^{2x} + c_2 x e^{2x}$$

$$y = c_1 e^{2x} + c_2 x e^{2x}$$

When $y = 3, \quad x = 0$

$$3 = c_1 + 0 \Rightarrow c_1 = 3$$

When $y' = 4, \quad x = 0$

$$y' = 2c_1 e^{2x} + c_2 (2x e^{2x} + e^{2x})$$

$$4 = 2(3)e^0 + c_2 (0 + e^0)$$

$$c_2 = -2$$

$$\therefore y = 3e^{2x} - 2xe^{2x}$$

Example5:

$$4y'' + 16y' + 17y = 0$$

$$y = 1, t = 0 \text{ and } y = 0, t = \pi$$

Solution:

$$(4D^2 + 16D + 17)y = 0$$

$$4m^2 + 16m + 17 = 0$$

$$m^2 + 4m = -\frac{17}{4}$$

$$m^2 + 4m + 4 = -\frac{17}{4} + 4$$

$$(m + 2)^2 = -\frac{1}{4}$$

$$m + 2 = \pm \frac{1}{2}i$$

$$m_1 = -2 + \frac{1}{2}i \text{ and } m_2 = -2 - \frac{1}{2}i$$

$$m_1 = -2 + \frac{1}{2}i \quad \text{and} \quad m_2 = -2 - \frac{1}{2}i$$

$$y = c_1 e^{(-2 + \frac{1}{2}i)t} + c_2 e^{(-2 - \frac{1}{2}i)t}$$

$$y = c_1 e^{-2t} e^{\frac{1}{2}it} + c_2 e^{-2t} e^{-\frac{1}{2}it}$$

$$y = e^{-2t} \left(A \cos \frac{1}{2}t + B \sin \frac{1}{2}t \right)$$

$$y = 1, \quad t = 0 \Rightarrow A = 1$$

$$y = 0, \quad t = \pi \Rightarrow B = 0$$

$$\therefore y = e^{-2t} \cos \frac{1}{2}t$$

The complete process for solving the homogeneous equation in all possible cases as following:

$$D.E. \quad ay'' + by' + cy = 0 \text{ or } (aD^2 + bD + c)y = 0$$

Characteristic equation:

$$am^2 + bm + c = 0 \quad \text{or} \quad aD^2 + bD + c = 0$$

Nature of the roots
of characteristic
equation

Condition on the
coefficients of the
characteristic
equation

Complete solution
of the differential
equation

Real and unequal

$$m_1 \neq m_2$$

$$b^2 - 4ac > 0$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Real and equal

$$m_1 = m_2$$

$$b^2 - 4ac = 0$$

$$y = c_1 e^{m_1 x} + c_2 x e^{m_2 x}$$

Conjugate complex

$$m_1 = p + iq$$

$$m_2 = p - iq$$

$$b^2 - 4ac < 0$$

$$y = e^{px} (A \cos qx + B \sin qx)$$

The nonhomogeneous equations

$$ay'' + by' + cy = f(x)$$

y = complementary function + particular integral

Theorem 4:

If Y_1 is a solution of $[ay'' + by' + cy = R_1(x)]$

and Y_2 is a solution of $[ay'' + by' + cy = R_2(x)]$

Then $Y = Y_1 + Y_2$ is a solution of

$$[ay'' + by' + cy = R_1(x) + R_2(x)]$$

Method of undetermined coefficients:

Example: $y'' + 4y' + 3y = 5e^{2x}$

Solution: Particular Integral

$$Y = Ae^{2x}$$

$$(4Ae^{2x}) + 4(2Ae^{2x}) + 3(Ae^{2x}) = 5e^{2x}$$

$$15A = 5 \Rightarrow A = \frac{1}{3}$$

$$\therefore \text{Particular Integral} = Y = \frac{1}{3}e^{2x}$$

Example: $y'' + 4y' + 3y = 5\sin 2x$

Solution: Particular Integral

$$Y = A\sin 2x + B\cos 2x$$

$$(-4A\sin 2x - 4B\cos 2x) +$$

$$4(2A\cos 2x - 2B\sin 2x)$$

$$+ 3(A\sin 2x + B\cos 2x) = 5\sin 2x$$

$$(-A - 8B)\sin 2x + (8A - B)\cos 2x = 5\sin 2x$$

$$-A - 8B = 5$$

$$8A - B = 0$$

Solve the above 2 Eqs. getting,

$$A = -\frac{1}{13} \quad \text{and} \quad B = -\frac{8}{13}$$

$$\therefore Y = -\frac{1}{13}\sin 2x - \frac{8}{13}\cos 2x$$

Example: $y'' + 3y' + 2y = 10e^{3x} + 4x^2$

Solution:

$$Y_1 = Ae^{3x}$$

$$Y_2 = Bx^2 + Cx + D$$

$$(9Ae^{3x}) + 3(3Ae^{3x}) + 2(Ae^{3x}) = 10e^{3x}$$

$$20Ae^{3x} = 10e^{3x}$$

$$A = \frac{1}{2}$$

$$2B + 3(2Bx + C) + 2(Bx^2 + Cx + D) = 4x^2$$

$$2B + 6Bx + 3C + 2Bx^2 + 2Cx + 2D = 4x^2$$

$$2Bx^2 + (6B + 2C)x + (2B + 3C + 2D) = 4x^2$$

$$2B = 4 \Rightarrow B = 2$$

$$6B + 2C = 0 \Rightarrow C = -6$$

$$2B + 3C + 2D = 0 \Rightarrow D = 7$$

$$Y = Y_1 + Y_2$$

$$Y = \frac{e^{3x}}{2} + 2x^2 - 6x + 7$$

∴ complete solution

$$y = c_1 e^{-2x} + c_2 e^{-x} + Y$$

$$\therefore y = c_1 e^{-2x} + c_2 e^{-x} + \frac{e^{3x}}{2} + 2x^2 - 6x + 7$$

Example: $y'' + 5y' + 6y = 3e^{-2x} + e^{3x}$

Solution:

$$m^2 + 5m + 6 = 0$$

$$(m + 2)(m + 3) = 0$$

$$m_1 = -2 \quad \text{and} \quad m_2 = -3$$

$$y = c_1 e^{-2x} + c_2 e^{-3x} + Y$$

$$Y = Y_1 + Y_2$$

$$Y_1 = A x e^{-2x}$$

$$Y_2 = B e^{3x}$$

$$Y = Axe^{-2x} + Be^{3x}$$

$$[4Axe^{-2x} - 2Ae^{-2x} - 2Ae^{-2x}] +$$

$$5[-2Axe^{-2x} + Ae^{-2x}] + 6[Axe^{-2x}] = 3e^{-2x}$$

$$4Axe^{-2x} - 4Ae^{-2x} - 10Axe^{-2x} + 5Ae^{-2x} + 6Axe^{-2x} = 3e^{-2x}$$

$$Ae^{-2x} = 3e^{-2x}$$

$$A = 3$$

$$[9Be^{3x}] + 5[3Be^{3x}] + 6[Be^{3x}] = e^{3x}$$

$$30Be^{3x} = e^{3x}$$

$$B = \frac{1}{30}$$

$$Y = 3xe^{-2x} + \frac{1}{30}e^{3x}$$

$$\therefore y = c_1e^{-2x} + c_2e^{-3x} + 3xe^{-2x} + \frac{1}{30}e^{3x}$$

Example: $y'' - 2y' + y = xe^x - e^x$

Solution: Homog.

$$m^2 - 2m + 1 = 0$$

$$(m - 1)^2 = 0$$

$$m_1 = m_2 = 1$$

$$y = c_1 e^x + c_2 x e^x + Y$$

$$Y = Y_1 + Y_2$$

$$Y = A_0 x^3 e^x + A_1 x^2 e^x$$

$$\therefore Y = (A_0 x^3 + A_1 x^2) e^x$$

$$\begin{aligned}
 &[A_0x^3 + (6A_0 + A_1)x^2 + (6A_0 + 4A_1)x + 2A_1]e^x \\
 &-2[A_0x^3 + (3A_0 + A_1)x^2 + 2A_1x]e^x \\
 &+[A_0x^3 + A_1x^2]e^x = xe^x - e^x
 \end{aligned}$$

$$6A_0xe^x + 2A_1e^x = xe^x - e^x$$

$$6A_0 = 1 \Rightarrow A_0 = \frac{1}{6}$$

$$2A_1 = -1 \Rightarrow A_1 = -\frac{1}{2}$$

$$Y = \left(\frac{1}{6}x^3 - \frac{1}{2}x^2\right)e^x$$

$$\therefore y = c_1e^x + c_2xe^x - \frac{1}{2}x^2e^x + \frac{1}{6}x^3e^x$$

Particular integrals by the method of variation of parametric:

$$ay'' + by' + cy = f(x) \quad (1)$$

$$\text{NonHomo. } y'' + P(x)y' + Q(x)y = R(x) \quad (2)$$

$$\text{Homo. Eq. } y'' + P(x)y' + Q(x)y = 0 \quad (3)$$

As we do in construction the complementary function, we attempt to find two functions of x , say u_1 and u_2 such that;

$$Y = u_1 y_1 + u_2 y_2$$

Will be a solution of nonhomogeneous equation (2)

Having two unknown functions, we required two equations for their determination:

Eq. (1) by substitution Y into given D.E. (2)

Eq. (2) remains at our disposal.

From

$$Y = u_1 y_1 + u_2 y_2$$

By differentiation ; we have:

$$Y' = (u_1 y_1' + y_1 u_1') + (u_2 y_2' + y_2 u_2')$$

Put;

$$u_1' y_1 + u_2' y_2 = 0 \quad (4)$$

Now,

$$Y' = u_1 y_1' + u_2 y_2'$$

And;

$$Y'' = (u_1 y_1'' + y_1' u_1') + (u_2 y_2'' + y_2' u_2')$$

Now, Substitution Y, Y' and Y'' into Eq. (2)

We obtain,

$$(u_1 y_1'' + y_1' u_1' + u_2 y_2'' + y_2' u_2') + P(x)(u_1 y_1' + u_2 y_2') + Q(x)(u_1 y_1 + u_2 y_2) = R(x)$$

Or;

$$u_1(y_1'' + P(x)y_1' + Q(x)y_1) + u_2(y_2'' + P(x)y_2' + Q(x)y_2) + (u_1' y_1' + u_2' y_2') = R(x)$$

$$\therefore y_1' u_1' + y_2' u_2' = R(x) \quad (5)$$

By solving Eqs. (4) and (5)

for u_1' and u_2' , we obtain,

$$\left. \begin{aligned} u_1' &= -\frac{y_2}{y_1 y_2' - y_2 y_1'} R(x) \\ u_2' &= \frac{y_1}{y_1 y_2' - y_2 y_1'} R(x) \end{aligned} \right\} \quad (6)$$

Example: $y'' - 4y = 3x$

Solution: Homog. $m^2 - 4 = 0$

$$m = \pm 2$$

$$y_h = c_1 e^{2x} + c_2 e^{-2x}$$

$$y = u_1(x)e^{2x} + u_2(x)e^{-2x} \quad u = f(x)$$

$$y' = 2u_1(x)e^{2x} + u_1'(x)e^{2x} - 2u_2(x)e^{-2x} + u_2'(x)e^{-2x}$$

$$y' = 2u_1(x)e^{2x} - 2u_2(x)e^{-2x} + u_1'(x)e^{2x} + u_2'(x)e^{-2x}$$

$$u_1'(x)e^{2x} + u_2'(x)e^{-2x} = 0 \quad (1)$$

$$y' = 2u_1e^{2x} - 2u_2e^{-2x}$$

$$y'' = 4u_1e^{2x} + 4u_2e^{-2x} + 2u_1'e^{2x} - 2u_2'e^{-2x}$$

Subst. y and y'' into D.E.

$$4u_1e^{2x} + 4u_2e^{-2x} + 2u_1'e^{2x} - 2u_2'e^{-2x} - 4(u_1e^{2x} + u_2e^{-2x}) = 3x$$

$$2u_1'e^{2x} - 2u_2'e^{-2x} = 3x \quad (2)$$

By Solving Eqs. (1) and (2) getting:

$$u_1' = \frac{3}{4}xe^{-2x} \quad \text{and} \quad u_2' = -\frac{3}{4}xe^{2x}$$

$$u_1 = -\frac{3}{8}xe^{-2x} - \frac{3}{16}e^{-ex} + c_1 \quad \text{and}$$

$$u_2 = -\frac{3}{8}xe^{2x} + \frac{3}{16}e^{ex} + c_2$$

Subst. u_1 and u_2 , get;

$$y = -\frac{3}{4}x + c_1e^{2x} + c_2e^{-2x}$$

Other method :

By formula

$$y_1 = e^{2x} \quad \text{and} \quad y_2 = e^{-2x}$$

$$u_1' = -\frac{y_2}{y_1 y_2' - y_2 y_1'} R(x)$$

$$u_2' = \frac{y_1}{y_1 y_2' - y_2 y_1'} R(x)$$

$$u_1' = - \frac{e^{-2x}}{e^{2x}(-2e^{-2x}) - (e^{-2x}(2e^{2x}))} * 3x$$

$$u_1' = - \frac{3xe^{-2x}}{-4}$$

$$= \frac{3}{4} xe^{-2x}$$

$$\therefore u_1 = -\frac{3}{4} xe^{-2x} - \frac{3}{16} e^{-2x}$$

$$u_2' = \frac{e^{2x}}{e^{2x}(-2e^{-2x}) - (e^{-2x}(2e^{2x}))} * 3x$$

$$u_2' = \frac{3xe^{2x}}{-4}$$

$$= -\frac{3}{4}xe^{2x}$$

$$\therefore u_2 = -\frac{3}{8}xe^{2x} + \frac{3}{16}e^{2x}$$

$$Y = u_1 y_1 + u_2 y_2$$

$$= \left(-\frac{3}{8} x e^{-2x} - \frac{3}{16} e^{-2x} \right) e^{2x}$$

$$+ \left(-\frac{3}{8} x e^{2x} + \frac{3}{16} e^{2x} \right) e^{-2x}$$

$$\therefore Y = -\frac{3}{4} x$$

$$y = y_{\text{homo}} + Y$$

$$\therefore y = c_1 e^{2x} + c_2 e^{-2x} - \frac{3}{4} x$$

Equations of higher order:

Example: $y''' + 3y'' + 3y' + y = 0$

Solution:

$$m^3 + 3m^2 + 3m + 1 = 0$$

$$(m + 1)^3 = 0$$

$$m_1 = m_2 = m_3 = -1$$

$$y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$$

Example: $(D^4 + 8D^2 + 16)y = 0$

Solution:

$$m^4 + 8m^2 + 16 = 0$$

$$(m^2 + 4)^2 = 0$$

$$m^2 = -4$$

$$m = \pm 2i, \pm 2i$$

$$y = c_1 \cos 2x + c_2 \sin 2x + c_3 x \cos 2x + c_4 x \sin 2x$$

Example: $y^{iv} - \lambda^4 y = 0$

Solution:

$$m^4 - \lambda^4 = 0$$

$$m^4 = \lambda^4$$

$$m^2 = \pm \lambda^2$$

$$m = \pm \lambda, \pm \lambda i$$

$$y = c_1 e^{\lambda x} + c_2 e^{-\lambda x} + c_3 \cos \lambda x + c_4 \sin \lambda x$$

Example: $y''' + 5y'' + 9y' + 5y = 3e^{2x}$

Solution: Homog.

$$m^3 + 5m^2 + 9m + 5 = 0$$

$$(m + 1)(m^2 + 4m + 5) = 0$$

$$m^2 + 4m + 5 = 0$$

$$(m + 2)^2 = -1$$

$$m + 2 = \pm i \Rightarrow m = -2 \pm i$$

$$m_1 = -1, \quad m_2 = -2 + i \text{ and } m_3 = -2 - i$$

$$y_{\text{homogeneous}} = c_1 e^{-x} + e^{-2x} (A \cos x + B \sin x)$$

$$y = c_1 e^{-x} + e^{-2x} (A \cos x + B \sin x) + Y$$

$$Y = Ae^{2x}$$

$$8Ae^{2x} + 5(4Ae^{2x}) + 9(2Ae^{2x}) + 5(Ae^{2x}) = 3e^{2x}$$

$$51Ae^{2x} = 3e^{2x} \Rightarrow A = \frac{1}{17} \Rightarrow \therefore Y = \frac{1}{17} e^{2x}$$

$$y = c_1 e^{-x} + e^{-2x} (A \cos x + B \sin x) + \frac{e^{2x}}{17}$$

Synthetic division ??????

Example $6y^{iv} + 7y''' - 13y'' - 4y' + 4y = 0$

$$6x^4 + 7x^3 - 13x^2 - 4x + 4 = 0$$

$$a: \quad \pm 1, \quad \pm 2, \quad \pm 4$$

$$b: \quad \pm 1, \quad \pm 2, \quad \pm 3, \quad \pm 6$$

$$\frac{a}{b} = \pm 1, \pm 2, \pm 4, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm \frac{2}{3}, \pm \frac{4}{3}$$

$$+1, -2, +\frac{1}{2}, -\frac{2}{3}$$

1	6	+7	-13	-4	+4
		+6	+13	0	-4
-2	6	13	0	-4	0
		-12	-2	+4	
1/2	6	+1	-2	0	
		+3	+2		
-2/3	6	+4	0		
		-4			
	6	0			

$$6x^4 + 7x^3 - 13x^2 - 4x + 4 = 0$$

$$6(x-1)(x+2)\left(x-\frac{1}{2}\right)\left(x+\frac{2}{3}\right) = 0$$

Example: $(D^4 + 8D^2 + 16)y = -\sin x$

Solution:

$$m^4 + 8m^2 + 16 = 0$$

$$(m^2 + 4)^2 = 0$$

$$m^2 = -4$$

$$m = \pm 2i, \pm 2i$$

$$y = c_1 \cos 2x + c_2 \sin 2x + c_3 x \cos 2x + c_4 x \sin 2x + Y$$

$$Y = A \cos x + B \sin x$$

$$(A \cos x + B \sin x) + 8(-A \cos x - B \sin x) +$$

$$16(A \cos x + B \sin x) = -\sin x$$

$$9A \cos x + 9B \sin x = -\sin x$$

$$A = 0 \quad \text{and} \quad B = -\frac{1}{9}$$

$$y = c_1 \cos 2x + c_2 \sin 2x + c_3 x \cos 2x + c_4 x \sin 2x - \frac{1}{9} \sin x$$

Operator Method:

$$\frac{dy}{dx} = Dy, \quad \frac{d^2y}{dx^2} = D^2y, \quad \frac{d^n y}{dx^n} = D^n y$$

Ex: find $(D^2 + 2D + 1)y$ If $y = x^2 + x$

$$\begin{aligned} D^2y + 2Dy + y &= 2 + 2(2x + 1) + x^2 + x \\ &= 4 + 5x + x^2 \end{aligned}$$

Define D^{-1}

Let $D^{-1} f(x) = u$

$$DD^{-1} f(x) = Du$$

$$\therefore Du = f(x)$$

$$\frac{du}{dx} = f(x)$$

$$u = \int f(x) dx$$

Theorem $f(D) = aD^2 + bD + c$

$$f(D)e^{mx} = am^2e^{mx} + bme^{mx} + ce^{mx}$$

$$f(D)e^{mx} = e^{mx}(am^2 + bm + c) = e^{mx}f(m)$$

$$(aD^2 + bD + c)y = e^{mx}$$

$$\therefore y = \frac{e^{mx}}{aD^2 + bD + c} \quad D = m$$

Ex : Solve

$$\frac{d^2 y}{dx^2} + y = 4e^{2x}$$

Solution: $(D^2 + 1)y = 4e^{2x}$

Homo. $D^2 + 1 = 0 \Rightarrow D = \pm i$

$$y_{\text{homo}} = c_1 \cos x + c_2 \sin x$$

$$y_p = \frac{4e^{2x}}{(2)^2 + 1} = \frac{4}{5} e^{2x}$$

$$\therefore y = c_1 \cos x + c_2 \sin x + \frac{4}{5} e^{2x}$$

For;

$$f(x) = \sin Bx \quad \text{or} \quad f(x) = \cos Bx$$

$$(aD^2 + bD + c)y = f(x)$$

$$y_p = \frac{f(x) \begin{bmatrix} \cos Bx \\ \sin Bx \end{bmatrix}}{aD^2 + bD + c}$$

Put: $D^2 = -B^2$

Ex : Solve,

$$(D^2 - 1)y = \sin 2x$$

Solution:

Homo. $D^2 - 1 = 0 \Rightarrow D = \pm 1$

$$y_{\text{homo}} = c_1 e^x + c_2 e^{-x}$$

$$y_p = \frac{\sin 2x}{D^2 - 1} = \frac{\sin 2x}{-2^2 - 1} = -\frac{1}{5} \sin 2x$$

$$\therefore y = c_1 e^x + c_2 e^{-2x} - \frac{1}{5} \sin 2x$$

Euler – Equation:

Try general form;

$$ax^2 \frac{d^2 y}{dx^2} + bx \frac{dy}{dx} + cy = f(x) \dots \dots (1) \quad 2^{nd} \text{ order}$$

Solution:

$$\text{Let } t = \ln x$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{1}{x} \times \frac{dy}{dt} \Rightarrow \frac{dy}{dt} = x \frac{dy}{dx}$$

$$\frac{d^2 y}{dx^2} = \frac{1}{x} \frac{d^2 y}{dt^2} \frac{1}{x} + \frac{dy}{dt} \left(-\frac{1}{x^2} \right)$$

$$\frac{d^2 y}{dx^2} = \frac{1}{x^2} \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right)$$

$$x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} - \frac{dy}{dt}$$

Subst. into Eq.(1)

$$a \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) + b \frac{dy}{dt} + cy = g(t)$$

$$a \frac{d^2 y}{dt^2} + (b - a) \frac{dy}{dt} + cy = g(t)$$

2nd order with coefficients

Example: Solve

$$x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} + y = x$$

Solution:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt}$$

$$\frac{dy}{dt} = x \frac{dy}{dx}$$

$$x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} - \frac{dy}{dt}$$

$$\left(\frac{d^2 y}{dt^2} - \frac{dy}{dt}\right) + 3\frac{dy}{dt} + y = e^t$$

$$\frac{d^2 y}{dt^2} + 2\frac{dy}{dt} + y = e^t$$

$$\text{Homog. } m^2 + 2m + 1 = 0$$

$$y_h = c_1 e^{-t} + c_2 t e^{-t}$$

$$y_p = \frac{e^t}{(1)^2 + 2(1) + 1} = \frac{1}{4} e^t$$

$$y = c_1 e^{-t} + c_2 t e^{-t} + \frac{e^t}{4}$$

$$y = \frac{c_1}{x} + c_2 \frac{\ln x}{x} + \frac{x}{4}$$

Applications:

Buckling of columns;

A slender column fails under compressive force when reach a certain value this is called "buckling load"

From strength of materials

$$\frac{d^2 y}{dx^2} = -\frac{M}{EI}$$

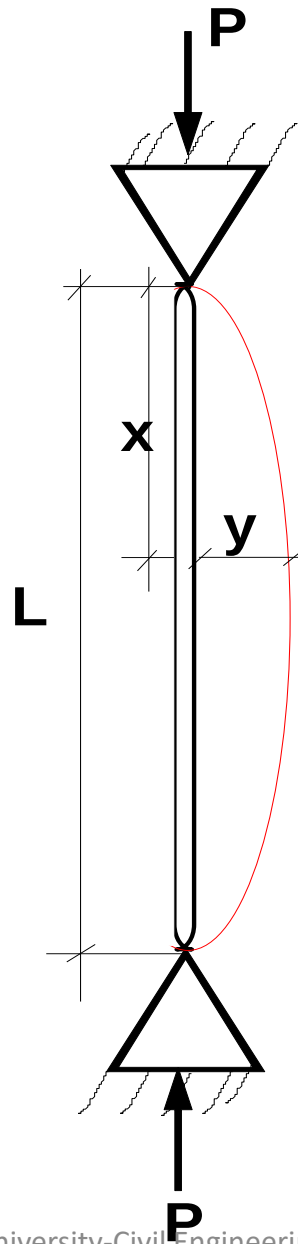
1– *Buckling of columns hinged at both ends;*

$$\frac{d^2 y}{dx^2} = -\frac{M}{EI}$$

At distance x , $M = Py$

$$\frac{d^2 y}{dx^2} = -\frac{Py}{EI}$$

$$\frac{d^2 y}{dx^2} + \frac{P}{EI} y = 0$$



$$\frac{P}{EI} = k^2$$

$$\frac{d^2 y}{dx^2} + k^2 y = 0$$

$$m^2 + k^2 = 0 \Rightarrow m = \pm ki$$

$$y = c_1 \cos kx + c_2 \sin kx$$

B.Cs.

$$y = 0 \text{ for } x = 0 \text{ and } x = L$$

$$y = 0 \text{ for } x = 0$$

$$0 = c_1 \cos 0 + 0 \Rightarrow c_1 = 0$$

$$y = c_2 \sin kx$$

$$y = 0 \text{ for } x = L$$

$$0 = c_2 \sin kL + 0 \Rightarrow c_2 \neq 0$$

$$\therefore \sin kL = 0 \Rightarrow kL = 0, \pi, 2\pi, \dots$$

$$kL = \pi \Rightarrow k^2 L^2 = \pi^2 \Rightarrow k^2 = \frac{\pi^2}{L^2}$$

$$kL = \pi \Rightarrow k^2 L^2 = \pi^2 \Rightarrow k^2 = \frac{\pi^2}{L^2}$$

$$\frac{P}{EI} = \frac{\pi^2}{L^2}$$

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

P_{cr} : *buckling load*

L : *length of column*

E : *modulus of elasticity*

I : *moment of inertia*

2 – Buckling of columns fixed at both ends;

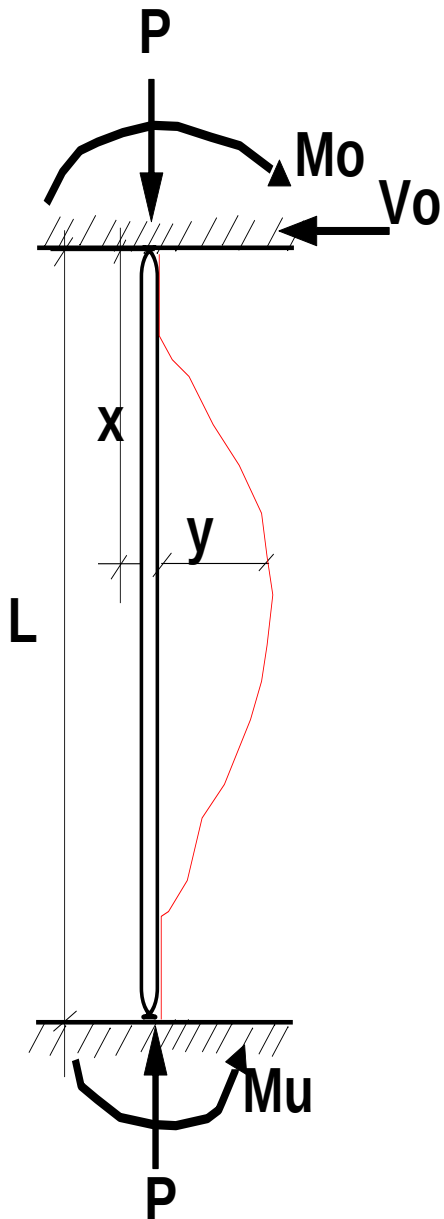
$$V_o = 0$$

$$M = Py - M_o$$

$$\frac{d^2 y}{dx^2} = -\frac{M}{EI}$$

$$\frac{d^2 y}{dx^2} = -\frac{(Py - M_o)}{EI}$$

$$\frac{d^2 y}{dx^2} + \frac{P}{EI} y = \frac{M_o}{EI}$$



$$\frac{P}{EI} = k^2$$

$$\frac{d^2 y}{dx^2} + k^2 y = \frac{M_o}{EI}$$

$$m^2 + k^2 = 0 \Rightarrow m = \pm ki$$

$$y_h = c_1 \cos kx + c_2 \sin kx$$

$$\frac{d^2 y}{dx^2} + k^2 y = \frac{M_o}{EI}$$

$$(D^2 + k^2)y = \frac{M_o}{EI}$$

$$(D^2 + k^2)y = \frac{M_o}{EI}$$

$$y_p = \frac{M_o e^0}{EI(D^2 + k^2)}$$

$$y_p = \frac{M_o}{EI(0 + k^2)} = \frac{M_o}{EI k^2} = \frac{M_o}{EI \left(\frac{P}{EI}\right)}$$

$$\therefore y_p = \frac{M_o}{P}$$

$$y = c_1 \cos kx + c_2 \sin kx + \frac{M_o}{P}$$

B.Cs.

$$y = 0 \text{ for } x = 0 \text{ and } x = L$$

$$\frac{dy}{dx} = 0 \text{ for } x = 0 \text{ and } x = L$$

$$\frac{dy}{dx} = -c_1 k \sin kx + c_2 k \cos kx$$

$$\frac{dy}{dx} = 0 \quad \text{for } x = 0$$

$$0 = 0 + c_2 k \cos kx \Rightarrow c_2 = 0$$

$$y = c_1 \cos kx + \frac{M_o}{P}$$

$$y = 0 \quad \text{for } x = 0$$

$$0 = c_1 (1) + \frac{M_o}{P} \Rightarrow c_1 = -\frac{M_o}{P}$$

$$y = -\frac{M_o}{P} \cos kx + \frac{M_o}{P}$$

$$\therefore y = \frac{M_o}{P} (1 - \cos kx)$$

$$y = 0 \quad \text{for } x = L$$

$$0 = \frac{M_o}{P} (1 - \cos kL)$$

$$\text{either } \frac{M_o}{P} = 0 \quad \text{or } 1 - \cos kL = 0$$

$$\therefore \cos kL = 1 \Rightarrow kL = 0, 2\pi, 4\pi, \dots$$

$$kL = 2\pi$$

$$k^2 L^2 = 4\pi^2 \Rightarrow \frac{P}{EI} L^2 = 4\pi^2$$

$$P_{cr} = \frac{4\pi^2 EI}{L^2} = \frac{\pi^2 EI}{\left(\frac{1}{2}L\right)^2}$$

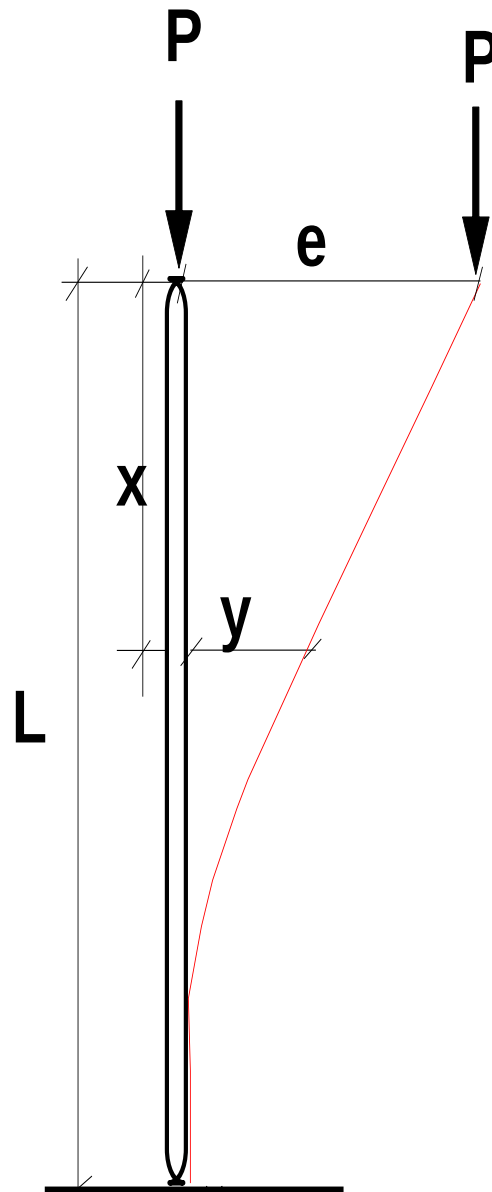
3– *Buckling of columns fixed at the bottom,
free at the top;*

$$M = -P(e - y)$$

$$\frac{d^2 y}{dx^2} = -\frac{M}{EI}$$

$$\frac{d^2 y}{dx^2} = \frac{P(e - y)}{EI}$$

$$\frac{d^2 y}{dx^2} + \frac{P}{EI} y = \frac{Pe}{EI}$$



$$\text{Let } \frac{P}{EI} = k^2$$

$$\frac{d^2 y}{dx^2} + k^2 y = k^2 e$$

$$m^2 + k^2 = 0 \Rightarrow m = \pm ki$$

$$y_h = c_1 \cos kx + c_2 \sin kx$$

$$y_p = A, \quad y_p'' = 0$$

$$0 + k^2 A = k^2 e \Rightarrow A = e$$

$$\therefore y_p = e$$

$$y = c_1 \cos kx + c_2 \sin kx + e$$

B.Cs.

$$y = 0 \quad \text{for } x = L$$

$$y = e \quad \text{for } x = 0$$

$$\frac{dy}{dx} = 0 \quad \text{for } x = L$$

$$y = c_1 \cos kx + c_2 \sin kx + e$$

$$y = e \quad \text{for } x = 0$$

$$e = c_1(1) + 0 + e \Rightarrow c_1 = 0$$

$$y = c_2 \sin kx + e$$

$$\frac{dy}{dx} = c_2 k \cos kx$$

$$\frac{dy}{dx} = 0 \quad \text{for } x = L$$

$$0 = c_2 k \cos kL$$

$$c_2 k \neq 0$$

$$\therefore \cos kL = 0 \Rightarrow kL = \frac{\pi}{2}$$

$$k^2 L^2 = \frac{1}{4} \pi^2$$

$$\frac{P}{EI} L^2 = \frac{1}{4} \pi^2$$

$$P_{cr} = \frac{EI \pi^2}{(2L)^2}$$